Calculus 1
Workbook Solutions

Formal definition of the limit

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MATH

ONE-SIDED LIMITS

■ 1. What is $\lim_{x \to -7^+} x^2\sqrt{x+7}$?

Solution:

$$\lim_{x \to -7^+} x^2\sqrt{x+7} = 0$$

x	-6.09	-6.9	-6.99	-6.999	-6.9999	-7
Value	35.38	15.056	4.886	1.5481	0.48999	0

■ 2. The graph of $f(x)$ is shown below. What does the graph show about the value of $\lim_{x \to -1^+} f(x)$?

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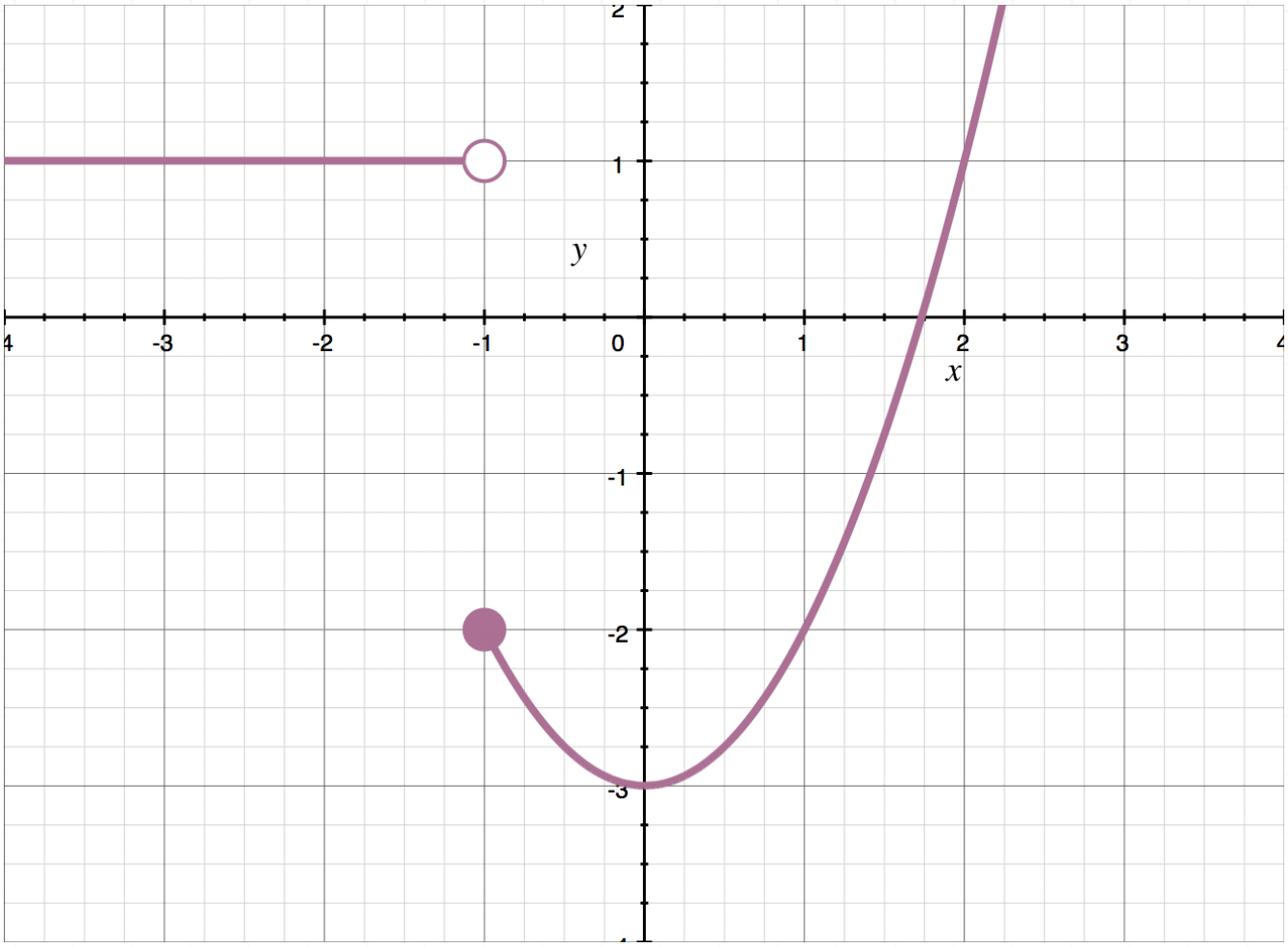

Solution:

From the graph, the limit is

$$\lim_{x \to -1^+} f(x) = -2$$

■ 3. The table shows values of $k(x)$. What is $\lim_{x \to -5^-} k(x)$?

x	-5.1	-5.01	-5.001	-5	-4.999	-4.99	-4.9
k(x)	-392.1	-3,812	-38,012	?	37,988	3,788	368.1

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Solution:

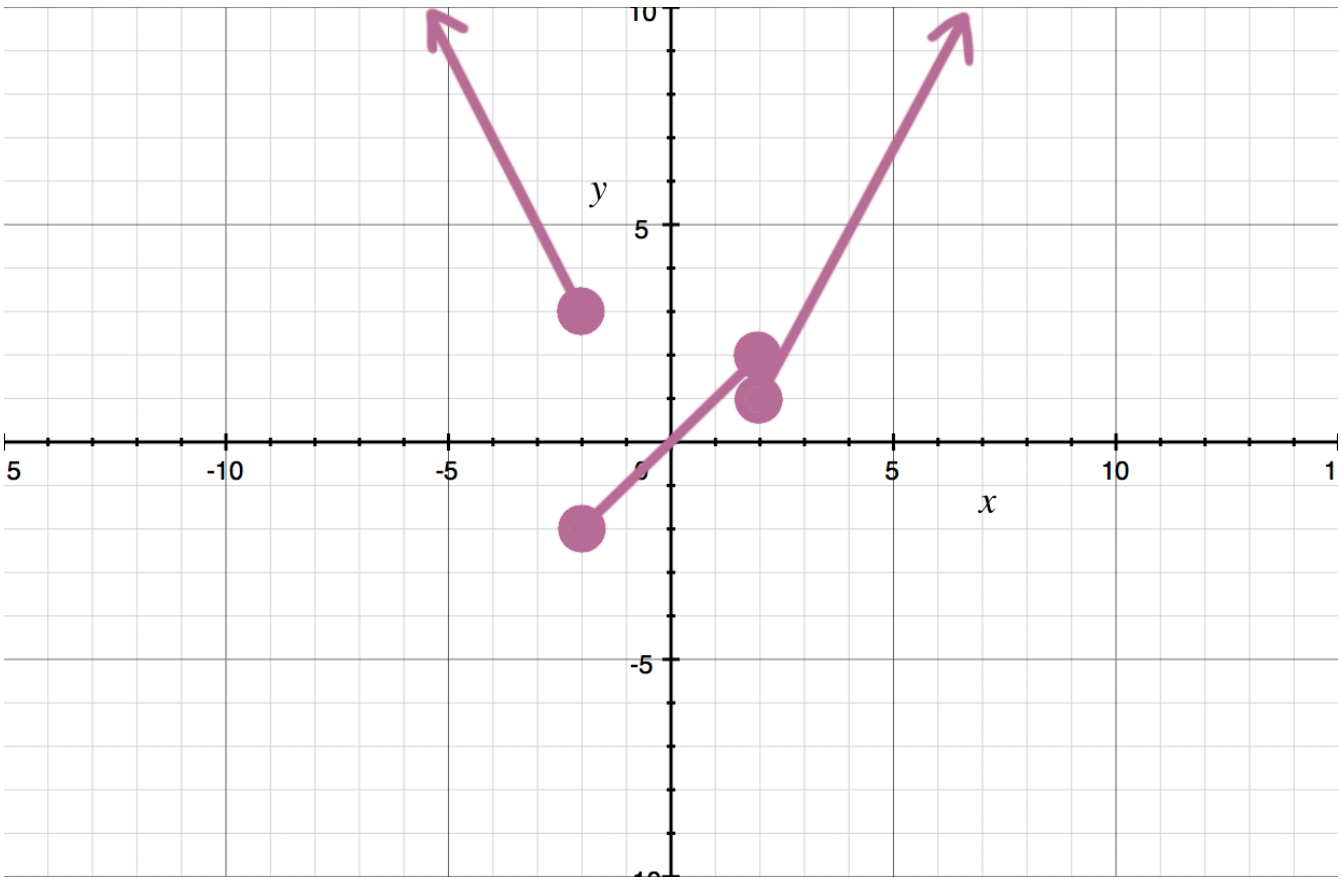
$$\lim_{x \to -5^-} k(x) = -\infty$$

■ 4. What is $\lim_{x \to -2^-} h(x)$?

$$h(x) = \begin{cases} -2x - 1 & x < -2 \\ x & -2 \leq x < 2 \\ 2x - 3 & x \geq 2 \end{cases}$$

Solution:

The graph of $h(x)$ is



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Based on the graph, the limit is 3. Or we could plug into the first piece of the function, which is the piece that approaches $x = -2$ from the left side.

$$\lim_{x \to -2^-} h(x) = \left[-2(-2) - 1\right] = 3$$

■ 5. What is $\lim_{x \to 6^+} g(x)$?

$$g(x) = \frac{x^2 + x - 42}{x - 6}$$

Solution:

We could tell that the limit is 13 by making a table,

x	6	6.001	6.01	6.1
g(x)	?	13.001	13.01	13.1

Alternatively, we could have factored the numerator, canceled like terms, and then evaluated at the limit.

$$g(x) = \frac{x^2 + x - 42}{x - 6}$$

$$g(x) = \frac{(x + 7)(x - 6)}{x - 6}$$

$$g(x) = x + 7$$

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Then the limit is

$$\lim_{x \to 6^+} x + 7$$

$$6 + 7$$

$$13$$

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PROVING THAT THE LIMIT DOES NOT EXIST

■ 1. Prove that the limit does not exist.

$$\lim_{x\to 0} \frac{-2|3x|}{3x}$$

Solution:

The left-hand limit is

$$\lim_{x\to 0^-} \frac{-2|3x|}{3x} = \frac{6x}{3x} = 2$$

The right-hand limit is

$$\lim_{x\to 0^+} \frac{-2|3x|}{3x} = \frac{-6x}{3x} = -2$$

Since the left- and right-hand limits aren't equal, the limit does not exist.

The graph of the function would also prove that the limit doesn't exist.

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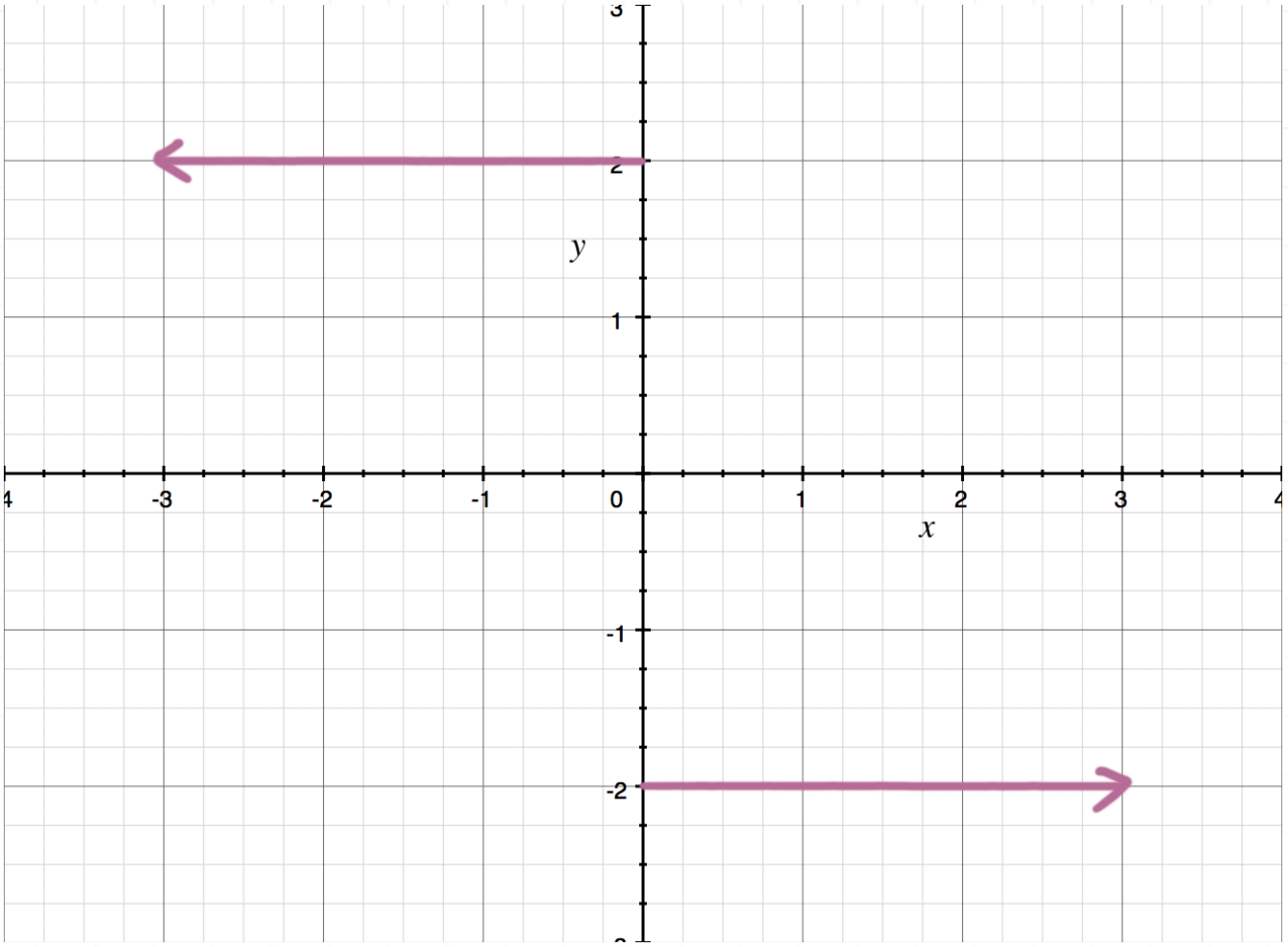

2. Prove that the limit does not exist.

$$\lim_{x\to -5} \frac{x^2+7x+9}{x^2-25}$$

Solution:

The left-hand limit is

$$\lim_{x\to -5^-} \frac{x^2+7x+9}{x^2-25} = -\infty$$

The right-hand limit is

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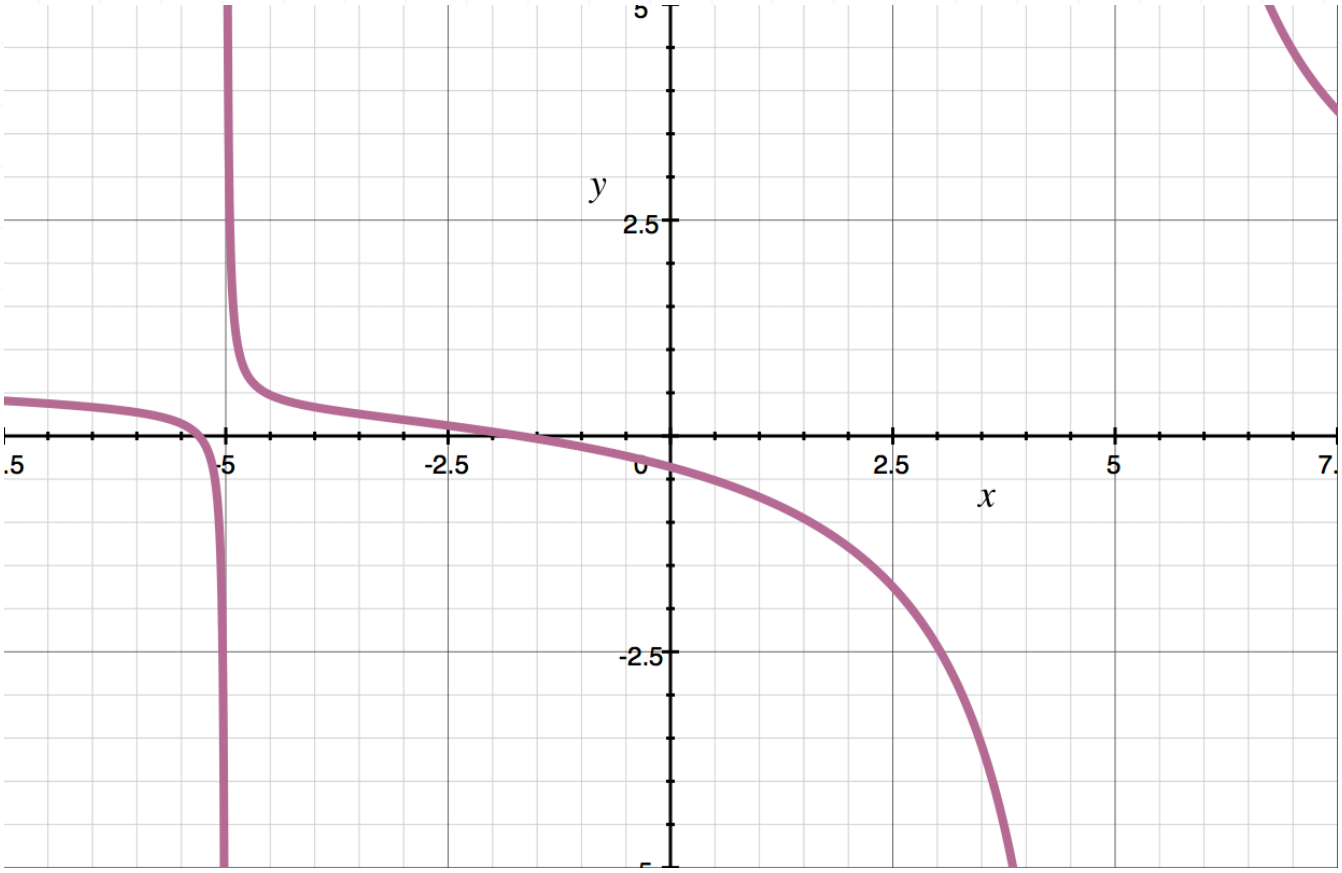
$$\lim_{x \to -5^+} \frac{x^2 + 7x + 9}{x^2 - 25} = \infty$$

Since the left- and right-hand limits aren't equal, the limit does not exist.

The graph of the function would also prove that the limit doesn't exist.



■ 3. Prove that $\lim_{x \to 1} f(x)$ does not exist.

$$f(x) = \begin{cases} -3x + 2 & x < 1 \\ 3x - 2 & x \geq 1 \end{cases}$$

Solution:

The left-hand limit is

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$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-3x+2) = \big[-3(1)+2\big] = -1$$

The right-hand limit is

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3x-2) = \big[3(1)-2\big] = 1$$

Because the left- and right-hand limits aren't equal, the limit does not exist.

$$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

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PRECISE DEFINITION OF THE LIMIT

■ 1. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x \to 4} 5x - 16 = 4$$

Solution:

If $0 < |x - 4| < \delta$, then $|(5x - 16) - 4| < \epsilon$. So,

$$|5x - 20| < \epsilon$$

$$|5(x - 4)| < \epsilon$$

$$|5| \cdot |x - 4| < \epsilon$$

$$5 \cdot |x - 4| < \epsilon$$

$$|x - 4| < \frac{\epsilon}{5}$$

Now if $|x - 4| < \epsilon/5$ and $0 < |x - 4| < \delta$, then if $\epsilon > 0$ then $\delta < \epsilon/5$. Therefore, the limit equation is true.

■ 2. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x \to -7} -2x + 15 = 29$$

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Solution:

If $0 < |x-(-7)| < \delta$ then $|-2x+15-29| < \epsilon$. Or we could rewrite this as $0 < |x+7| < \delta$ and $|-2x-14| < \epsilon$. So,

$$|(-2)(x+7)| < \epsilon$$

$$|-2| \cdot |x+7| < \epsilon$$

$$2 \cdot |x+7| < \epsilon$$

$$|x+7| < \frac{\epsilon}{2}$$

Now if $|x+7| < \epsilon/2$ and $0 < |x+7| < \delta$, then if $\epsilon > 0$ then $\delta < \epsilon/2$. Therefore, the limit equation is true.

■ 3. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x \to 16} \left(\frac{2}{5}x - \frac{17}{5} \right) = 3$$

Solution:

If $0 < |x-16| < \delta$ then $\left| \left((2/5)x - (17/5) \right) - 3 \right| < \epsilon$. Or we could rewrite this as $0 < |x-16| < \delta$ and

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$$\left|\left(\frac{2}{5}x-\frac{17}{5}\right)-\frac{15}{5}\right|<\epsilon$$

$$\left|\frac{2}{5}x-\frac{32}{5}\right|<\epsilon$$

$$\left|\frac{2}{5}(x-16)\right|<\epsilon$$

$$\left|\frac{2}{5}\right||x-16|<\epsilon$$

$$|x-16|<\frac{5}{2}\epsilon$$

Now if $|x-16|<(5/2)\epsilon$ and $0<|x-16|<\delta$, then if $\epsilon>0$, then $\delta<(5/2)\epsilon$.

Therefore, the limit equation is true.

■ 4. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x\to 7}\left(\frac{x^2-15x+56}{x-7}\right)=-1$$

Solution:

We'll apply the precise definition to the given limit.

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If $0 < |x-7| < \delta$, then $\left| \left(\frac{x^2-15x+56}{x-7} \right) - (-1) \right| < \epsilon$.

If $0 < |x-7| < \delta$, then $\left| \left(\frac{x^2-15x+56}{x-7} \right) - \frac{-1(x-7)}{x-7} \right| < \epsilon$.

So,

$$\left| \left(\frac{x^2-15x+56}{x-7} \right) + \frac{x-7}{x-7} \right| < \epsilon$$

$$\left| \frac{x^2-14x+49}{x-7} \right| < \epsilon$$

$$\left| \frac{(x-7)(x-7)}{x-7} \right| < \epsilon$$

$$|x-7| < \varepsilon$$

Now, if $|x-7| < \epsilon$ and $0 < |x-7| < \delta$, then if $\epsilon > 0$ and $\delta < \epsilon$. Therefore, the limit equation is true.

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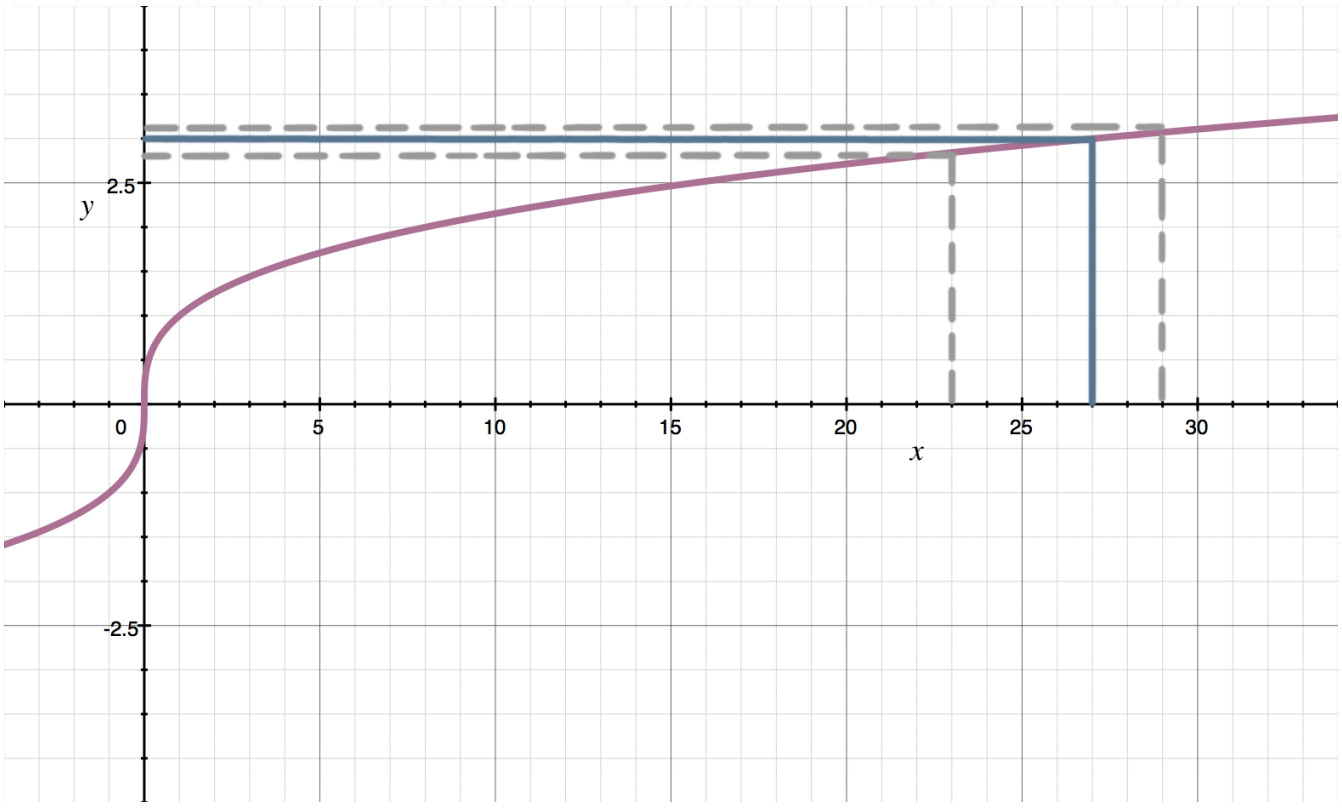
PRECISE DEFINITION OF THE LIMIT, FINDING DELTA

■ 1. Use the graph of $f(x) = \sqrt[3]{x}$ and the precise definition of the limit to find δ, such that if $|x - 27| < \delta$ then $\left|\sqrt[3]{x} - 3\right| < 0.1$.



Solution:

Use the general form of the precise definition of the limit.

If $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

So if $|x - 27| < \delta$, then $\left|\sqrt[3]{x} - 3\right| < 0.1$. Use the smaller δ or ϵ that will be greater than 0.1

$$3.1 = \sqrt[3]{x}$$

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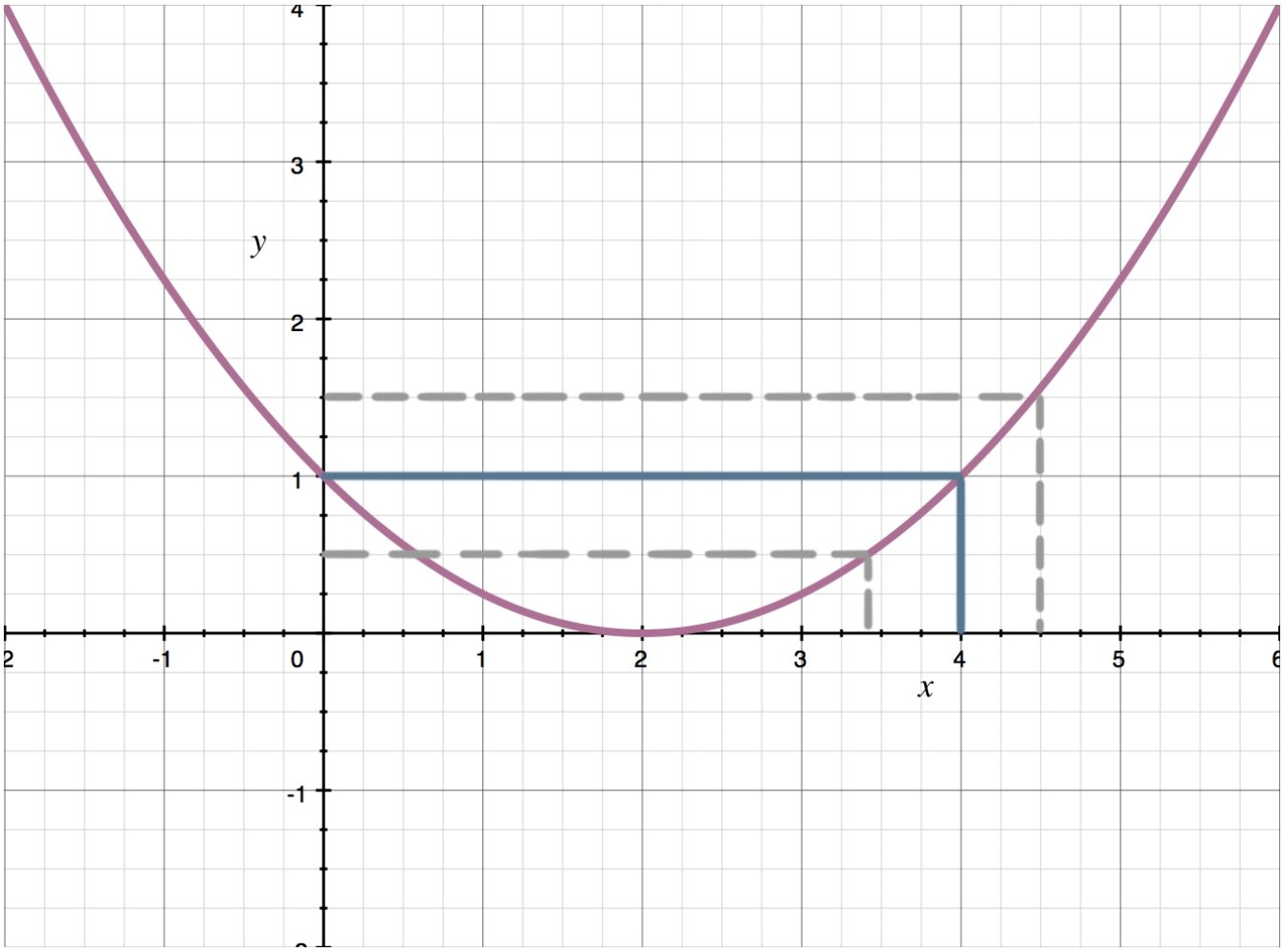
$$(3.1)^3 = x$$

$$x = 29.719$$

Therefore, $\delta = 29.719 - 27 = 2.791$.

■ 2. Use the graph of the given function and the precise definition of the limit to find δ, such that if $|x - 4| < \delta$ then $|g(x) - 1| < 0.5$.

$$y = \frac{1}{4}x^2 - x + 1$$



Solution:

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Use the general form of the precise definition of the limit.

If $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$

So if $|x - 4| < \delta$, then

$$\left| \left(\frac{1}{4}x^2 - x + 1 \right) - 1 \right| < 0.5$$

$$\left| \frac{1}{4}x^2 - x \right| < 0.5$$

So

$$1.5 = \frac{1}{4}x^2 - x$$

$$x^2 - 4x - 6 = 0$$

$$x = 2 \pm \sqrt{10}$$

or

$$0.5 = \frac{1}{4}x^2 - x$$

$$x^2 - 4x - 2 = 0$$

$$x = 2 \pm \sqrt{6}$$

The smallest δ is given by $x = 2 + \sqrt{6}$, so that's the one we need to use.

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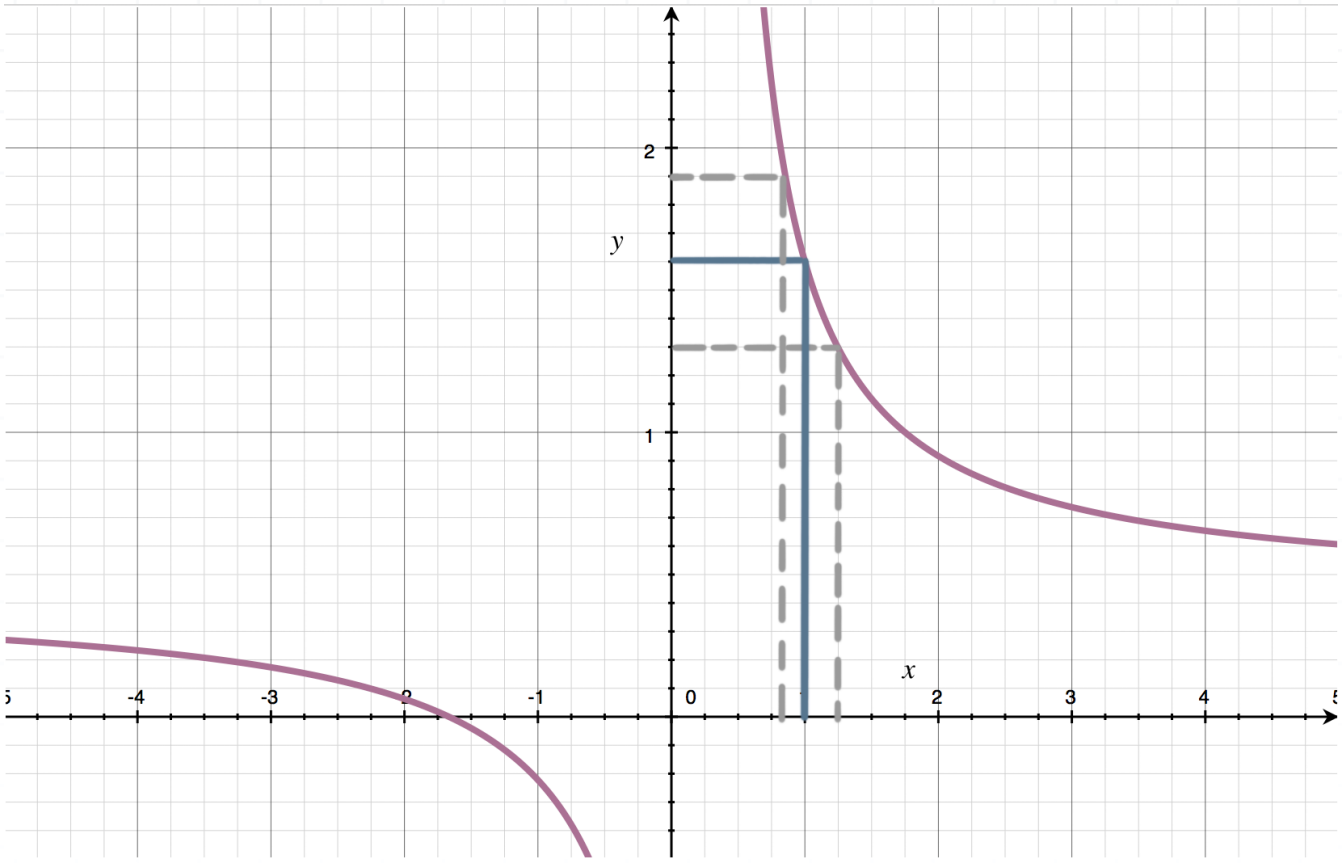
■ 3. Use the graph of the given function and the precise definition of the limit to find δ, such that if $|x-1|<\delta$ then $|h(x)-1.6|<0.3$.

$$h(x)=\frac{3x+5}{7x-2}$$



Solution:

Use the general form of the precise definition of the limit.

If $0<|x-a|<\delta$, then $|f(x)-L|<\epsilon$

So if $|x-1|<\delta$, then

$$\left|\left(\frac{3x+5}{7x-2}\right)-1.6\right|<0.3$$

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So

$$1.9 = \frac{3x+5}{7x-2}$$

$$x = 0.8544$$

or

$$1.3 = \frac{3x+5}{7x-2}$$

$$x = 1.2459$$

Therefore, $\delta = 1 - 0.8544 = 0.1456$ or $\delta = 1.2459 - 1 = 0.2459$. Use the smaller δ or ϵ that will be greater than 0.3. Therefore, $\delta = 0.1456$.

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